

Space-time generated from a set of binary units

P. Peretto^a

CEN Grenoble, CNRS/LPM2C, BP 166, 38042 Grenoble cedex 9, France

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Abstract. This paper shows that the properties of space-time that constitutes the background of the theory of special relativity, namely its dimensionality, the correct partition of dimensions between one time-type and three space-type dimensions and the Minkowski metrics, may emerge from a set of completely interacting binary units structured by a noise defined in a Landau-type free energy of Higgs fields and by gauge symmetries, in particular those related to the permutation group of four objects.

1 Introduction

Lloyd has put forward the idea that the universe could be entirely made up of elementary binary physical systems, to which we shall refer as “cosmic bits” (or CB’s) and that the laws of physics could be seen as a series of computations carried out on those bits. Relying upon entropy calculations Lloyd suggested that the number of bits is of the order of [1, 2]

$$N_{\text{CB}} \approx 10^{120}.$$

We may find this idea quite logical: the conventional approach in physics has been reductionist, explaining “complex” phenomena by the means of more “simple” systems. Introducing cosmic bits leads to push this approach to its limit since it is impossible to define physical systems with less than one bit of information.

The price to pay is the loss of familiar notions such as space, time, mass . . . , that constitute our environment. It is not possible either to define the shape of a CB, not even its position, since this would require the introduction of more information to characterize the CB, which is not allowed. The notion of space-time loses its ontological status and must be rebuilt from a set of axioms that are defined in Sect. 2.

A physics based on CB’s is a priori neither a relativistic (or classical) physics nor a quantum physics but it is a combinatorial and statistical physics. In actual fact the status of CB’s may be seen as classical since the states σ_a and σ_b of two CB’s are commuting real numbers

$$\sigma_a \sigma_b = \sigma_b \sigma_a \quad \sigma_\alpha = \pm 1$$

but one may build, by using the CB’s, physical objects such as fields, that do not commute (see Sect. 6).

The idea that space-time might be discrete is not a new one. It has already been mentioned in antiquity to provide an answer to the Zeno’s paradox according to which

Achille can never catch a tortoise [3]. In 1887 Lord Kelvin considered a discrete foam-like model for the aether. In 1928 Einstein studied a combinatorial model of space with the hope of unifying electromagnetism and gravitation [4]. Quantum field theorists were, in the sixties, faced with serious difficulties with infinite quantities and it was suggested that the existence of some cut-off associated with the discrete nature of space-time could remove the divergences. The success of renormalization techniques, however, made the hypothesis of discrete spaces unnecessary.

The idea of a discrete space-time was not completely abandoned anyway, and Penrose, for example, proposed that space-time could be constituted of spin networks [5]. The resulting model is called the spin-foam model of space-time [6] (the term “foam” has been introduced by Wheeler). Hawking proposed another type of foam, namely a foam of mini black holes that continuously form and dissolve [7].

These foam theories assume that the structures supporting the fundamental physical systems, namely spins or mini black holes, are based either on quantum physics, as is the case for spins, or on the usual space-time structure of general relativity, as is the case for the mini black holes (black holes also display a quantum behaviour since, according to Hawking, they may be considered as black bodies).

The model we put forward in this article is not based on such presuppositions. It assumes that space is entirely, and only, made of CB’s that is to say, of physical systems whose states are fully determined by a binary variable. Cosmic bits can be seen as Ising spins (commuting variables indeed). All CB’s interact through random binary interactions (see Fig. 1).

Our model can therefore be seen as a variety of spin glass models [8]. If it differs from the above mentioned foam models, it adopts, however, some of their views. An entropy is associated with black holes, for example, that is to say, black holes contain a given, finite, amount of information bits. A space made of mini black holes therefore is such that

^a e-mail: Secr.lpmmc@grenoble.cnrs.fr

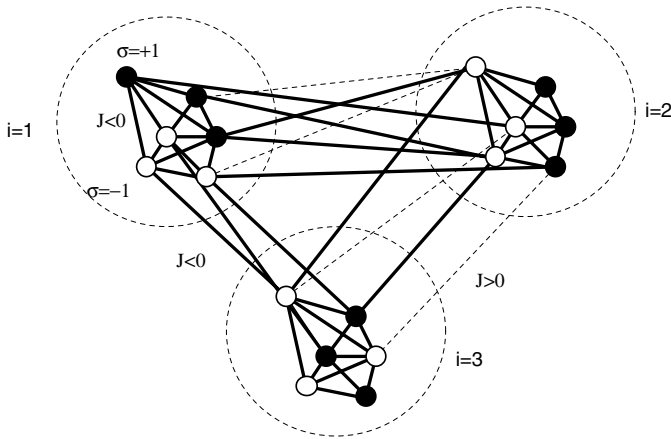


Fig. 1. A spin glass model for the physical space: a set of cosmic bits in states $\sigma = +1$ (bulk circles) or $\sigma = -1$ (in open circles) interconnected through negative interactions (bulk lines) and positive interactions (dotted lines). The cosmic points are sets of cosmic bits fully interconnected through negative interactions (gathered within the large dotted circles). All lines of the figures are only relational. They have no geometrical meanings

any piece of space houses a finite amount of information. This hypothesis has been considered by some theorists as a sufficient condition for the space to be discrete. We also agree with Penrose that particles are not actors created in a pre-existing space-time background but rather that CB's, which are the fundamental components of particles, create their own space-time and define their own geometries.

The purpose of this article is to show that the very simple model we propose can account for the structural properties of space-time as we know them. To achieve this goal, the strategy consists in showing that the axioms determine a dynamics of states described by a Klein-Gordon equation. The identification of constants which enter the definition of the Klein-Gordon equation with their expressions in terms of the three parameters of the model then leads to a possible physical interpretation of these fundamental constants.

2 A model for space

The four axioms of the model are the following:

- (i) The cosmos, space as a whole, is entirely made of a countable set of cosmic bits (CB's) and the state of the universe, $\psi = \{\sigma_a\}$ $a = 1, 2, \dots, N_{CB}$, is determined by a family $\sigma_a = \pm 1$ of CB's analogous to Ising spins
- (ii) The realizable states of the universe are those which minimize the following quadratic real functional of CB states

$$H(\psi) = \sum_{ab} J_{ab} \sigma_a \sigma_b = \psi^T J \psi \quad (1)$$

where the elements of the matrix J can only take two values:

$$J_{ab} = \pm J/n$$

To support this statement we may argue that we know, by experience, that the universe is not in a state of complete disorder and therefore that all the possible states of

the universe are not realized. As a consequence the model assumes that there must exist a functional of bits states that is, at least approximately, minimized for the realizable physical states of the universe. The most general functional is written as an expansion over all possible multiplets of CB's.

$$H(\psi) = \sum_{\text{multiplets } ab\dots c} \sum J_{ab\dots c} \sigma_a \sigma_b \dots \sigma_c .$$

Actually all terms are not necessary because it is always possible to reproduce the minima by $H(\psi) = \sum_{ab} J_{ab} \sigma_a \sigma_b = \psi^T J \psi$ using only quadratic terms.

As an example let us assume that the physical state

$$\psi_0 = \{\xi_a\}$$

is a minimum for the most general functional. This is also the physical state that minimizes the quadratic functional with interactions given by

$$J_{ab} = -\xi_a \xi_b$$

Moreover the CB's cannot be distinguished from one another. The same holds for their interactions because a given set of different interactions could be used to discriminate one bit from the others. All interactions have therefore been given the same absolute value.

(iii) A "cosmic point" i (or CP) is a set of CB's fully interconnected through negative binary (*ferromagnetic*) interactions, $J_{ab} = -J/n$, where n is the number of CB's inside a CP. A CB is a close neighbor of all other bits of the CP that it belongs to and the notion of distance is meaningless inside a CP. The set of CP's constitutes a partition of the universe. It is assumed that n , the number of CB's in a CP, and N , the number of CP's in the universe, are huge numbers. Since space is homogeneous, all CP's play exactly the same role. In particular n must be approximately the same for all CP's. The fluctuations of n are statistical and are given by

$$\Delta n/n = 1/(n)^{1/2} .$$

They approach zero in the thermodynamic limit of very large n 's.

(iv) The bits of a CP are subject to a degree of disorder whose strength is given by a parameter b called "cosmic noise". b may be considered as the inverse of a temperature and the larger b the smaller the temperature. This temperature is similar to the one introduced by Higgs in his definition of the Higgs field [9]. It facilitates the application of statistical mechanics to derive the thermodynamic properties of cosmic points.

Let us give a few arguments to support this axiom. In order to give masses to fermions, Higgs introduced a scalar field h whose value is obtained by minimizing a Landau-type free energy

$$F(h) = \frac{\lambda}{2} h^2 + \frac{\mu}{4} h^4 \quad \text{with} \quad \lambda = b_c - b \quad , \quad \mu > 0 \quad (2)$$

where b is the inverse of a temperature. If $b < b_c$ then $h = 0$, and vacuum is called symmetrical vacuum, whereas

if $b > b_c$ the Higgs field does not vanish and the vacuum is asymmetrical which is a necessary condition to give mass to fermions. One generally believes that introducing the parameter b is just a mathematical trick, but we propose here to take it seriously and to consider a CP as a thermodynamic system which will be studied by using the tools of statistical mechanics. This is not a trivial assertion, because statistical mechanics relies upon two fundamental hypotheses that we must discuss. On the one hand it assumes that there exists a reservoir and, on the other hand, that the ergodic hypothesis is satisfied, namely that temporal averages may be replaced by ensemble averages.

(1) A CP, that is to say a set of bits all interacting through negative interactions, is linked to other CP's through a sum of binary interactions with random signs. The result is a generally weak CP to CP interaction obeying a Gaussian distribution. One therefore may consider that the set of all other CP's constitute a reservoir for a given cosmic point. (2) The validity of the ergodic hypothesis seems to be more difficult to assess because time is not defined inside a CP. Actually this makes the ergodic hypothesis even more attractive, since, in the absence of any concept of time, only ensemble averages may be given a physical meaning.

The functional H_i of a cosmic point is given by

$$H_i = -\frac{J}{n} \sum_{\langle \alpha\beta \rangle} \sigma_{i\alpha} \sigma_{i\beta}$$

where 'i' is the label of a CP and α is the label of a CB inside the CP. The sum is taken over all pairs of cosmic bits inside the cosmic point 'i'. It is an extensive quantity since, if all bits are such that $\sigma_{i\alpha} = +1$ or $\sigma_{i\alpha} = -1$, one has

$$H_i \simeq -nJ$$

and the functional is proportional to the size n of the system. Then, the distribution of cosmic points states is Maxwellian:

$$\rho(H_i) = \frac{1}{Z_i} \exp(-bH_i)$$

where Z_i is the partition function

$$Z_i = \sum_{\{\sigma_{i\alpha}\}} \exp(-bH_i).$$

The sum is over all possible realizations of cosmic bit states for CP 'i'. The statistical mechanics of fully connected spin glasses, systems close to the one introduced in this paper, has been worked out by Sherrington and Kirkpatrick [10].

To summarize, our model of space, defined as a set of randomly interacting binary units, is determined by three parameters, respectively the interaction J between the CB's, the number n of CB's into a CP and the cosmic noise b .

3 Order parameters and gauge symmetries of cosmic points

Order, called a polarization, tends to spontaneously appear in an isolated CP (a CP isolated from the other CP's) due

to a balance between the tendency of CB's to take the same value $\sigma = \pm 1$, that arises from the ferromagnetic interactions, and a tendency to disorder that results from the action of the thermal noise. The polarization of a given CP is the thermal average of an order parameter defined by

$$s_i = \frac{1}{n} \sum_{\alpha} \sigma_{i\alpha} \quad \alpha = 1, \dots, n; \quad i = 1, \dots, N.$$

The polarization is given by the following mean field equation:

$$\langle s_i \rangle = \varphi_i = \tanh(bJ\varphi_i), \tag{3}$$

an exact equation in the limit of large n 's.

This equation is obtained by minimizing the "free functional" given by (see Appendix A)

$$F(\varphi_i) = n \left(\frac{-J}{2} \varphi_i^2 + \frac{1}{b} \left[\left(\frac{1+\varphi_i}{2} \right) \text{Ln}(1+\varphi_i) + \left(\frac{1-\varphi_i}{2} \right) \text{Ln}(1-\varphi_i) \right] \right). \tag{4}$$

Expanding the logarithmic functions to the fourth order yields

$$F(\varphi_i) = \lambda \varphi_i^2 + \mu \varphi_i^4 \tag{2'}$$

with

$$\lambda = \frac{n(1-bJ)}{2b}; \quad \mu = \frac{n}{12b}. \tag{5}$$

Equation (3) has non zero solutions, called asymmetrical vacuum, if $\lambda < 0$, that is to say if $bJ > 1$. The polarization is a scalar field whose value is determined by the label 'i'. Since it obeys the Higgs field formalism (in particular it is a scalar field and it minimizes the same Landau type "free functional"), it is tempting to see the polarization as being the Higgs field itself. An asymmetrical vacuum is a necessary condition for gauge theories such as the GSW (Glashow, Salam, and Weinberg) theory [9] to be relevant. Therefore one must have $bJ > 1$. Let us also recall that, in mean field theories, all polarization fluctuations vanish.

To introduce the concept of dimensions we now try to answer the following question: can a cosmic point be considered as a set of d subsets (sub-cosmic points so to speak) such that the system obtained by putting these d sub-cosmic points together, reproduces the polarization of the cosmic point?

Let $n_{i\mu}$ (with $\mu = 1, \dots, d$) be the number of bits associated with the sub-cosmic point μ of CP 'i'. For the time being there is no restriction on the values of the $n_{i\mu}$ except that they have to obey the following constraint:

$$\sum_{\mu} n_{i\mu} = n.$$

The order parameter of the sub-cosmic point μ is defined by

$$s_{i\mu} = \frac{1}{n_{i\mu}} \sum_{\alpha} \sigma_{i\mu\alpha},$$

where α is here the label of a CB belonging to the sub-cosmic point μ in CP ‘ i ’. The polarization of an isolated sub-cosmic point μ is given by an equation similar to (3) with a renormalized interaction

$$\langle s_{i\mu} \rangle = \tanh \left(b \frac{J n_{i\mu}}{n} \langle s_{i\mu} \rangle \right). \quad (6)$$

This polarization does not vanish if

$$bJ \frac{n_{i\mu}}{n} > 1. \quad (1)$$

Then all sub-cosmic points are polarized at once, whatever the $n_{i\mu}$ ’s, if the following condition is fulfilled:

$$bJ \sum_{\mu} \frac{n_{i\mu}}{n} = bJ > \sum_{\mu} 1 = d. \quad (7)$$

This is a necessary condition for the set of sub-cosmic points to be considered as a CP. The condition is not sufficient, however, since in the range

$$1 < bJ < \frac{n}{n_{i\mu}},$$

the sub-cosmic point μ is not polarized. In fact we have only considered so far non interacting sub-cosmic points. The sub-cosmic points do interact and these are the interactions that stabilize the global polarization of a CP as a whole.

To study those interactions it is necessary to look carefully at the thermal properties of the set of sub-cosmic points. The calculation of the free functional of the set of polarizations is carried out in Appendix A. We obtain

$$\begin{aligned} F(\{\langle s_{i\mu} \rangle\}) &= \frac{-J}{2n} \sum_{\mu\nu} n_{i\mu} n_{i\nu} \langle s_{i\mu} \rangle \langle s_{i\nu} \rangle \\ &+ \sum_{\mu} \frac{n_{i\mu}}{2b} \\ &\times \left((1 + \langle s_{i\mu} \rangle) \text{Ln}(1 + \langle s_{i\mu} \rangle) \right. \\ &\quad \left. + (1 - \langle s_{i\mu} \rangle) \text{Ln}(1 - \langle s_{i\mu} \rangle) \right). \quad (8) \end{aligned}$$

The polarizations are given by the minima of this free functional. With $d = 1$, (8) becomes identical to (4).

A state of the universe can then be reformulated in terms of polarizations. It is written as a column vector

$$\psi = \begin{pmatrix} \vdots \\ \phi_i \\ \vdots \\ \vdots \end{pmatrix}$$

where the state ϕ_i of CP ‘ i ’ is defined by

$$\phi_i = \begin{pmatrix} \vdots \\ \varphi_{i\mu} \\ \vdots \\ \varphi_{id} \end{pmatrix}$$

and the *polarization components* by

$$\varphi_{i\mu} = \frac{n_{i\mu}}{n} \langle s_{i\mu} \rangle. \quad (9)$$

The polarization components must obey the following constraint

$$\sum_{\mu} \varphi_{i\mu} = \frac{1}{n} \sum_{\mu} n_{i\mu} \langle s_{i\mu} \rangle = \frac{1}{n} \left\langle \sum_{\mu\alpha} \sigma_{i\mu\alpha} \right\rangle = \varphi_i. \quad (10)$$

In our model, physics is determined by the polarization components but the labels of the components are arbitrary and, therefore, relabeling the names of the components must not change the physical phenomena.

This gives rise to a first local gauge invariance (GI1) which states that physics must be left invariant under the operations of the group S_d of permutations of d objects. Locality means that this invariance is associated with one cosmic point, an object with no internal dimensions.

In other respects, (10) may be seen as the equation of a hyperplane in a d -dimensional space to which we shall refer as a *representation space*. A state of a CP is a point in this representation space and GI1 is a symmetry that permutes the coordinates of the point.

This is not the only gauge symmetry, however, because nothing determines the orientation of axes in the representation space whatsoever. Therefore physics must also be left invariant under the operations of group $SO(d)$. This symmetry forces the polarization components to obey a constraint obtained from (10) by averaging over all possible rotations. Let us write (10) as

$$\varphi_i^2 = \sum_{\mu} \varphi_{i\mu}^2 + \sum_{\mu, \nu \neq \mu} \varphi_{i\mu} \varphi_{i\nu}.$$

Averaging this expression over all possible rotations makes the cross terms vanish and the constraint is

$$\sum_{\mu} \varphi_{i\mu}^2 = \varphi_i^2. \quad (11)$$

This is the expression of a second local gauge invariance (GI2).

Another way to obtain the expression of the free functional is to look at the thermal average of the functional (1) in the framework of the mean field theory. Equation (1) is written as

$$H = \psi^T J \psi = \sum_{i\mu\alpha, j\nu\beta} J_{i\mu\alpha, j\nu\beta} \sigma_{i\mu\alpha} \sigma_{j\nu\beta}$$

and, since there is no polarization fluctuation in mean field theories, the free functional can be expressed as

$$\begin{aligned} F = \langle H \rangle &= \sum_{i\mu\alpha, j\nu\beta} J_{i\mu\alpha, j\nu\beta} \langle \sigma_{i\mu\alpha} \rangle \langle \sigma_{j\nu\beta} \rangle \\ &= \sum_{i\mu, j\nu} \left(\sum_{\alpha\beta} J_{i\mu\alpha, j\nu\beta} \right) n_{i\mu} \langle s_{i\mu} \rangle n_{j\nu} \langle s_{j\nu} \rangle \end{aligned}$$

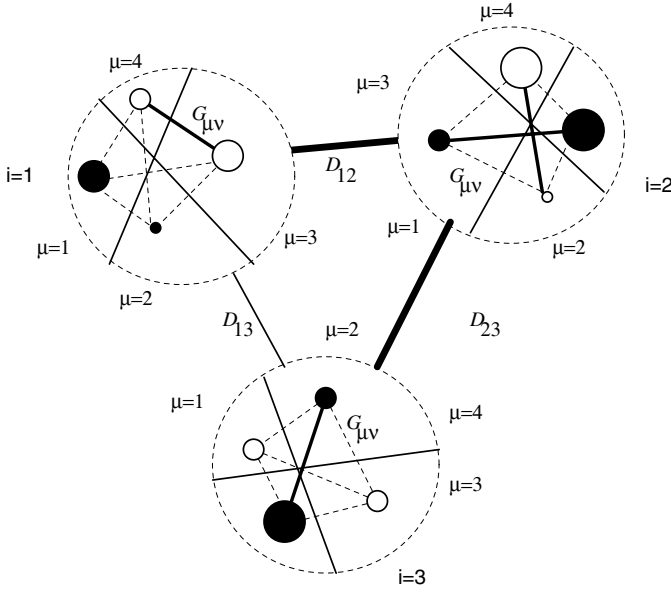


Fig. 2. The same model, as in Fig. 1, once the thermal averages have been taken into account and the cosmic points divided in a series of $d=4$ sub-cosmic points. Open and bulk circles represent polarization components

$$= \sum_{i\mu, j\nu} K_{i\mu, j\nu} \varphi_{i\mu} \varphi_{j\nu}$$

with

$$K_{i\mu, j\nu} = \sum_{\alpha\beta} J_{i\alpha\mu, j\beta\nu}.$$

Due to gauge invariances GI1 and GI2 these parameters factorize

$$K_{i\mu, j\nu} = \Delta_{ij} G_{\mu\nu}$$

that is to say

$$K = \Delta \otimes G$$

since, for any given pair of CP's 'i' and 'j', they must be left unchanged whatever the permutations or rotations of polarization components (see Fig. 2).

G is a square, real, symmetric, d -dimensional matrix that operates on the polarization components of a given CP. $G_{\mu\nu}$ describes the interaction between the μ and the ν components of the polarization inside one and the same CP.

Δ is a square, real, symmetric, N -dimensional matrix. Its element Δ_{ij} describes the interaction that links the CP 'i' to the CP 'j'. Δ_{ij} is a sum of n binary random variables. It is therefore a random variable whose distribution is Gaussian and centered at zero.

4 The realizable states as elements of vector spaces

In this section we show that the set of realizable states of a given matter field makes a Hilbert space.

The realizable states are those that minimize the free functional

$$F = \psi^T \Delta \otimes G \psi \tag{12}$$

under the set of constraints GI2 (11)

$$\phi_i^T \phi_i = \varphi_i^2.$$

The problem can be solved by using the method of Lagrange multipliers which requires one to minimize the following expression

$$\sum_{ij\mu\nu} \varphi_{i\mu} \Delta_{ij} G_{\mu\nu} \varphi_{j\nu} - \sum_i \kappa_i \left(\sum_{\mu} \varphi_{i\mu}^2 - \varphi_i^2 \right)$$

that, therefore, must be left invariant under the alteration of one component, that is to say

$$2\delta\varphi_{i\mu} \left(\sum_{j\nu} \Delta_{ij} G_{\mu\nu} \varphi_{j\nu} - \kappa_i \varphi_{i\mu} \right) = 0.$$

The polarization components therefore obey an eigenvalue equation:

$$\sum_{j\nu} \Delta_{ij} G_{\mu\nu} \varphi_{j\nu} = \kappa_i \varphi_{i\mu}. \tag{13}$$

Space, in the present approach, is homogeneous and the eigenvalues are the same whatever the cosmic point. The problem is then equivalent to minimizing (12) under the constraint

$$\psi^T \psi = \sum_i \varphi_i^2 = C(bJ).$$

Its solution is

$$(\Delta \otimes G) \psi = \kappa \psi. \tag{14}$$

The set of eigenstates corresponding to a given eigenvalue κ is called a *matter field*. The eigenstates associated with a given eigenvalue κ , that is to say with a given matter field, constitutes a vector space, since any linear combination of two eigenstates is also an eigenstate

$$(\Delta \otimes G) (\lambda_1 \psi_1 + \lambda_2 \psi_2) = \kappa (\lambda_1 \psi_1 + \lambda_2 \psi_2).$$

Moreover, this vector space may be endowed with an inner product since

$$\psi^T \psi = \sum_i \phi_i^T \phi_i = \sum_i \varphi_i^2 = C^t.$$

The vector space spanned by the states of a matter free field (identified by the eigenvalue κ), is then an Hilbert space.

Several remarks can be made:

(i) The eigenspaces are highly degenerate because they account for all realizable states of a given matter field.

Since there are so many possible states for a matter field and since the matter field must be in one of these states, the realized state must be chosen at random.

(ii) We will see that the indices μ and ν are space labels as well as time labels. Since time and space indices are intertwined in the eigenvalue equation (14), an eigenstate describes a state of the universe for all times and at all places at once (a complete history so to speak).

(iii) The eigenstates may be normalized as long as the polarization of CP's is non-zero, that is to say as long as $bJ > 1$. It is then, so to speak, impossible to "compress" the eigenstates.

5 Dimension organization

It must, first of all, be stressed that all considerations that we develop in this section are made in the framework of a physics which is still combinatorial and, therefore, that time, space, dimensions, dimensionality, ... are nothing but names that illustrate the objects we introduce. The identification with the usual meanings of those terms will be carried out in Sect. 6.

We have seen that the description of a CP as a set of sub-cosmic points loses its meaning if the sub-cosmic points are not all polarized at once. We have seen that, for this to occur, it is necessary that the condition (7) be satisfied. Given bJ , this condition yields a highest value for d

$$d = \text{Int}(bJ) \quad (15)$$

d is called the *dimensionality* of space and its experimental value settles a range of values for the parameter bJ

$$d < bJ < d + 1.$$

This section is devoted to a more careful study of the matrix G .

G is a square matrix of order d . Gauge invariance GI1 states that G must be left unchanged under the operations of the permutation group S_d of d objects. G may be written accordingly as

$$G_{\mu\nu} = J_0 \delta_{\mu\nu} + J_1 (1 - \delta_{\mu\nu}). \quad (16)$$

Then, the functional of a given CP can be written as

$$F(\phi_i) = J_0 \sum_{\mu} \varphi_{i\mu}^2 + J_1 \sum_{\mu\nu(\neq\mu)} \varphi_{i\mu} \varphi_{i\nu}. \quad (17)$$

To express the parameters J_0 and J_1 in terms of b , J , and n , the three parameters of the model, it is necessary to identify (17) with (8). By expanding the logarithmic functions to the second order and by using the definition (9) of polarization components, (8) becomes

$$F(\phi_i) = n \left[\sum_{\mu} \left(\frac{-J}{2} + \frac{1}{2b} \frac{n}{n_{i\mu}} \right) \varphi_{i\mu}^2 - \frac{J}{2} \sum_{\mu\nu(\neq\mu)} \varphi_{i\mu} \varphi_{i\nu} \right].$$

The ratios $n/n_{i\mu}$ are of the order of d and the identification of (17) with (8) yields

$$J_0 \cong -\frac{n}{2} \left(J - \frac{d}{b} \right),$$

$$J_1 \cong -\frac{n}{2} J.$$

If a convenient form of G is (16), this form is not unique because any unitary transformation of this representation is also convenient, in particular the one that diagonalizes G . Since

$$\text{Det}(G - \lambda I) = (J_0 + (d-1)J_1 - \lambda)(J_0 - J_1 - \lambda)^{d-1},$$

the diagonal representation identifies two and only two subspaces for G . The first one corresponds to the eigenvalue

$$G_{tt} = J_0 + (d-1)J_1 = \frac{nd}{2} \left(\frac{1}{b} - J \right)$$

and is not degenerate. This subspace, of dimension 1 whatever d , will be called *time dimension*. The other subspace corresponds to the eigenvalue

$$G_{rr} = J_0 - J_1 = \frac{nd}{2b}.$$

This subspace, of dimension $d-1$, corresponds to *space dimensions*.

The dimensionality of our space is $d=4$ which implies $4 < bJ < 5$. We have, therefore, $bJ > 1$ and vacuum is asymmetric.

In 4-dimensional spaces there is one time dimension with eigenvalue

$$G_{tt} = 2n \left(\frac{1}{b} - J \right)$$

and three equivalent space dimensions with eigenvalue

$$G_{rr} = \frac{2n}{b}.$$

Let us write

$$G_{\mu\mu} = \text{Sign}(G_{\mu\mu}) |G_{\mu\mu}| = \varepsilon_{\mu} G_{\mu} \quad (18)$$

and define the metric tensor g by $g_{\mu\nu} = \delta_{\mu\nu} \varepsilon_{\mu}$. Since $bJ > 1$ its elements are

$$g_{tt} = \text{Sign} \left(\frac{nd}{2b} (1 - bJ) \right) = -1,$$

$$g_{rr} = \text{Sign} \left(\frac{nb}{2b} \right) = +1,$$

that is to say

$$g = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & \ddots & \\ & & & +1 \end{pmatrix}$$

and the metrics is Minkowskian. It is worth pointing out that there is no more ambiguity on the sign of g (whereas relativistic mechanics does not distinguish between g and $-g$). The three dimensions of space and the time dimension constitute a conformal space with dilatation factors given by G_r and G_t , respectively.

As a matter of fact, this organization of space is fully determined by the irreducible representations of groups of permutations of d objects. For example G , in a four dimensional space, is diagonalized along the following direct sum of irreducible representations of S_4 , the group of permutations of four objects (see Appendix B):

$$\Gamma_4 = \Gamma_1 \oplus \Gamma_3$$

a sum of two irreducible representations of dimensions 1 and 3 respectively.

6 Vector fields dynamics

The polarization of cosmic points constitutes, as we have seen, a scalar field which is possibly identified with the Higgs fields but the physical fields one observes in nature are 4-dimensional vector fields. They are defined as vectors in the representation space and their components are thus linear combinations of polarization components:

$$\psi_{i\mu} = \sum_{\nu} C_{\mu\nu} \varphi_{i\nu}, \quad (19)$$

where C is a d -dimensional matrix that operates in the internal (representation) space of CP's. It can be shown that the permutation gauge symmetry GI1 defines and organizes the various types of vector fields because this gauge invariance compels the matrix C to transform according to a direct sum of irreducible representations of S_4 . We shall not go farther in that direction, however, because this subject is outside the scope of the article.

We now consider the dynamics of vector fields.

Let us recall that Δ is a symmetric square matrix of order N and that an element Δ_{ij} of Δ represents the overall interaction between the CP's ' i ' and ' j '. This interaction is the result of a sum of randomly distributed binary interactions between the CB's which belong to the points. They are therefore random parameters obeying a centered Gaussian distribution. According to the LDU theorem of Banachiewicz [12], any square matrix can be expressed as a product of a lower triangular matrix L , of a diagonal matrix A and of an upper triangular matrix U . When the matrix is real and symmetric, as is the case for Δ , the two triangular matrices are each other transpose:

$$\Delta = D^T A D$$

where D is triangular, that is to say $D_{ij} = 0$ for $i < j$. D^T is the transpose matrix and A is diagonal. More precisely, since space is homogeneous, A is a spherical matrix (it is proportional to the unity matrix: $A_{ij} = a\delta_{ij}$) and one may take $a = 1$. Hence

$$\Delta = D^T D$$

and the realizable physical states are given by the following eigenvalue equation:

$$(D^T D) \otimes G \psi = \kappa \psi. \quad (20)$$

Just as the elements of Δ do, the elements of D obey a centered Gaussian distribution.

In our model, we may imagine that space is a sort of wall made with bricks, that are the CP's, binded one to the other by a cement which is provided by the elements of the matrix D .

To be more precise, let us consider the set of links D_{ij} associated with a given CP ' i '. This set has N elements (that is to say the number of CP's). An element is the sum of n randomly distributed elementary interactions $\pm J/n$. Since n is very large the (random) amplitudes of most elements are close to zero, of the order of or less than $\sigma = Jn^{-1/2}$. Since N is very large, however, it is also likely that some

elements are far away in the tails of the Gaussian distribution say 10σ or so. One may call this set of exceptional CP's the *neighborhood* of CP ' i '. This definition has, in the present context, no topological meaning whatsoever. The links between the neighbors make so far a web that can be embedded in spaces of any geometry.

One defines the increment of a polarization component μ of CP ' i ' by

$$\delta\varphi_{i\mu} = \sum_j D_{ij} \varphi_{j\mu},$$

that is to say

$$\delta\phi_i = \sum_j D_{ij} \phi_j.$$

This quantity is mainly determined by the neighborhood of CP ' i '. It is a small quantity indeed since

$$\sum_j D_{ij} \cong 0.$$

The increment of the μ th component of the vector field along dimension ν is written as

$$\delta_{\nu} \psi_{i\mu} = C_{\mu\nu} \delta\varphi_{i\nu} = C_{\mu\nu} \sum_j D_{ij} \varphi_{j\nu}$$

and one defines the partial derivatives of the matter field components as

$$\partial_{\nu} \psi_{i\mu} = \frac{C_{\mu\nu}}{l^*} \sum_j D_{ij} \varphi_{j\nu}$$

for the first order derivatives, and as

$$\partial_{\nu}^2 \psi_{i\mu} = \frac{C_{\mu\nu}}{l^{*2}} \sum_{jk} D_{ij}^T D_{jk} \varphi_{k\nu} \quad (21)$$

for second order derivatives. l^* is a parameter, called the *metric limit*, that defines the size of a CP. The metric limit l^* , if this notion has any meaning, is certainly smaller than the range we can reach with the available particles colliders and is probably much larger than the Planck's length.

One entry of (20) reads

$$\sum_{jk,\nu} D_{ij}^T G_{\mu\nu} D_{jk} \varphi_{k\nu} = \kappa \varphi_{i\mu}.$$

One solution of this equation is a set of polarization components. Therefore it is this equation that determines the repartition of the numbers $n_{i\mu}$ between the various sub-cosmic points.

With eqs. (18–21), this equation becomes

$$G_{\nu} l^{*2} g_{\nu} \partial_{\nu}^2 \psi_{i\mu} = \kappa C_{\mu\nu} \varphi_{i\nu}. \quad (22)$$

The summation over index ν is then carried out on both members of (22). By introducing the Minkowski metrics ($\varepsilon_t = -1$, $\varepsilon_r = +1$) one finds

$$G_r l^{*2} \Delta \psi_{\mu} - \left[G_t l^{*2} \frac{\partial^2 \psi_{\mu}}{\partial t^2} \right] = \kappa \psi_{\mu} \quad (23)$$

where Δ is the usual three dimensional Laplacian. Here we introduce two important parameters. The first one, called the “*velocity of light*”, is defined by

$$c = \left(\frac{G_r}{G_t} \right)^{1/2} \quad (24)$$

and the other one, called “*Planck’s constant*”, by

$$\hbar = l^* (G_r)^{1/2} .$$

The quotes are in order since, up to now, the identification of (23) with the usual Klein-Gordon equation of quantum mechanics is only formal.

Furthermore we write

$$\kappa = (mc)^2 .$$

With these definitions (24) becomes

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta + \left(\frac{mc}{\hbar} \right)^2 \right) \psi_\nu(r, t) = 0 \quad (25)$$

which is recognized a set of four Klein-Gordon equations indeed. This result strongly suggests that the present approach could give rise to an interpretation of quantum mechanics, but the link is not really achieved here. It remains to see how the various fields, boson and fermion fields, come into play and, a point of paramount importance, how they obey the relevant quantum statistics.

Let us look at the plane waves solutions of the Klein-Gordon equation

$$\psi(r, t) = \exp(i(kr - \omega t)) .$$

Substituting in (25) leads to

$$\left[-\frac{\omega^2}{c^2} + k^2 + \left(\frac{mc}{\hbar} \right)^2 \right] \psi(r, t) = 0$$

and therefore to

$$\hbar^2 \omega^2 = \hbar^2 c^2 k^2 + m^2 c^4 .$$

By defining $E = \hbar \omega$ (the Planck relation) and $p = \hbar k$ (the De Broglie relation) one finds

$$E^2 = c^2 p^2 + m^2 c^4$$

which is the formula of the energy of a free particle of mass m in special relativity. The Klein-Gordon equation plays a central role in physics: on the one hand its factorization leads to the Dirac equation whose symmetry fits the fermions symmetry; on the other hand, in the limit of small momenta, the Klein-Gordon equation reduces to the Schrödinger equation.

We have thus achieved a substantial part of our program but some important points still need to be addressed. A central issue is that a comprehensive theory of space-time must also provide an interpretation of quantum mechanics. Another point, that is probably difficult to accept, is to see

that the interactions between neighboring CP’s propagate the polarization components, that is to say propagate the matter fields, throughout the whole network of CP’s and that it is the process that builds space as a whole. It is worth noting that space is the same whatever the building field, be it a boson or a fermion field and, to use the metaphor we have introduced above, the wall is the same whatever the bricklayer. Would fields not exist, space would not exist either, which fits the Penrose point of view (Sect. 1): it is meaningless to imagine space as preexisting to fields and to see the network of CP’s as a very involved structure embedded in this space. Moreover, since the indices $\mu\nu$ refer to internal spaces of CP’s and since physics is to be left invariant either under the elements of S_4 or under the elements of $SO(4)$, there is nothing such as a parallelism between the internal dimensions of CP’s.

7 Conclusion

The purpose of this paper is to put forward the thesis that our physical space could be entirely made of binary units and to show that this structure may provide a new approach to understand the space and time of special relativity.

The model is based upon a two level organization.

(1) The fundamental level is that of “*cosmic bits*” (or CB’s), which are physical binary systems similar to Ising spins. The CB’s interact one with the other through random binary interactions (a spin glass model).

(2) The other level is that of “*cosmic points*” (or CP’s). A CP is a set of cosmic bits fully interconnected through ferromagnetic interactions. In a cosmic point, an entity analogous to the Einstein point of the Universe (or world point), the notions of space and time are meaningless.

The properties of the model are determined by three parameters:

- (a) the interaction J/n between the CB’s,
- (b) the number n of CB’s into a CP, and
- (c) a cosmic noise parameter b .

Let us summarize the main results derived in the article.

First of all the model introduces rather naturally a number of physical concepts that usually have an ontological status. This is the case for words such as dimensionality, dimension, space, time, matter fields etc. The theory strives to give a precise meaning to all those terms. It shows in particular the following.

- (a) The dimensionality d of space is given by

$$d = \text{Int}(Jb)$$

and, since the experimental value of d is $d = 4$, that one has $4 < Jb < 5$.

(b) The dimensions are organized along two, and only two, types of dimensions. On the one hand there is one, and only one, time-type dimension whatever d and, on the other hand, there are $d - 1$ equivalent space-type dimensions. The dimensionality of spatial dimensions in our universe is therefore $d - 1 = 3$.

- (c) The metrics is Minkowskian.

(d) The set of realizable states for a given matter free field

constitutes a Hilbert space.

(e) The dynamics of matter fields is determined by a Klein-Gordon-type equation.

(f) The two fundamental constants, the “velocity of light” and the “Planck’s constant”, can be expressed in terms of the parameters of the model.

For example the “velocity of light” is given by

$$c^2 = \frac{G_r}{G_t} = \frac{nd/2b}{nd/2(J-1/b)} = \frac{1}{bJ-1} \quad (26)$$

It is, in fact, a dimensionless, universal, parameter that determines the ratio between the standards of length and time.

Moreover the “Planck’s constant” is given by

$$\hbar = l^* G_r^{1/2} = l^* (n/2b)^{1/2} .$$

The “Planck’s constant” depends on the metric scale l^* whereas the “velocity of light” is independent on that scale. We have no precise idea of the value of the metric scale but, if the model has any meaning, it must be in the range

$$10^{-18} \text{cm} > l^* > 10^{-33} \text{cm}$$

where the lower limit is the Planck’s length and the upper limit is the length accessible to the present particles colliders. We also may argue that, since the model tentatively establishes a bridge between boson and fermion particles, the metric scale could possibly be the scale where the supersymmetric theories (SUSY) become accessible that is to say

$$l^* \cong 10^{-21} \text{cm} .$$

The energies corresponding to this metric scale are of the order of 10 000 TeV. They are so large that laboratory experimental observations are probably impossible. The model, however, could be relevant in the fields of high energy physics, of astrophysics or even of cosmology. Let us give an example in high energy physics and another in cosmology.

We have seen that c , the so-called “velocity of light”, provides a natural scale factor between the time-like component and the three space-like components of quadrivectors in the usual space-time. This remark may have a direct application in electroweak theory.

According to the GSW theory, the electromagnetic and weak interactions transform together according to the symmetry obtained by the product of $U(1)$, the gauge symmetry group associated with the electromagnetic interaction and $SU(2)$ the gauge symmetry Lie group associated with the weak interaction.

$$T_{EW} = U(1) \times SU(2) .$$

Since the generators of the product T_{EW} of Lie groups $U(1)$ and $SU(2)$ is given by the sum of their generators, the interaction is written as

$$\Theta = gW^0\sigma_0 + g' \sum_{\mu=1\dots3} W^\mu\sigma_\mu$$

where the identity matrix $\sigma_0 = 1^{(2)}$ is associated with $U(1)$ and the three Pauli matrices $\sigma_{x,y,z}$ with $SU(2)$. This expression defines four boson fields W . The indices μ are Lorentz

indices and, therefore, the four 2-dimensional matrices may be considered as a quadrivector. It is then convenient to assume that $g'/g=c$.

The Weinberg angle is defined by

$$\tan(\theta_W) = g'/g .$$

The experimentally accessible parameter is [see (26)]

$$\begin{aligned} \sin^2(\theta_W) &= \frac{\tan^2(\theta_W)}{1 + \tan^2(\theta_W)} = \frac{c^2}{1 + c^2} = \frac{1/(bJ-1)}{1 + 1/(bJ-1)} \\ &= \frac{1}{bJ} . \end{aligned}$$

Since $4 < bJ < 5$ one has

$$0.20 < \sin^2(\theta_W) < 0.25 .$$

The experimental value is

$$\sin^2\theta_W = 0.231$$

which is consistent with the prediction. With this value as a datum one has

$$bJ = 4.33$$

and

$$c = 0.548 .$$

This parameter provides a natural link between the standard unit of time (*sut*) and the standard unit of length (*sul*). Let us assume that 1 *sut* = 1 sec which fits our biological ranges of time. Then

$$1 \text{ sul} = c \times 2.99 \times 10^8 \text{ m} = 1.64 \times 10^8 \text{ m}$$

a standard of length of little practical interest for daily life. The situation is less uncomfortable in astronomy. Let us take the period of rotation of large planets (such as Jupiter) as the standard unit of time: 1 *sut* = 10 hours. Then the size of the solar system is of the order of 1 *sul*. As another example we take the period of rotation of large planets around the Sun as the standard unit of time: 1 *sut* = 10 years. Then the interstellar distances is of the order of 1 *sul*.

Finally let us consider an example in cosmology. When, in our model, the cosmic noise b is lower than its critical value $b = 1/J$ neither space-time nor fields or matter would exist. It would therefore be natural to consider the Big Bang as a phase transition where the cosmic noise crosses its critical value. Then inflation, a theory proposed by Guth [13] to account for the homogeneity of the primitive Universe despite the limited speed of light that would have hindered any communications between its various parts, could receive a simple explanation. In the vicinity of the critical value, when the Big Bang occurs, our model predicts that the “velocity of light” is infinite (see (26)) which would settle the causality problem.

More generally high energy physics and cosmology are the two fields where the present theory could bring about some more useful information.

The model, for example, could possibly give an internal structure (inside CP's) to quantum particles. If the case arises the present approach could also explain the structure of fermion families in the Standard Model. It is even possible that it could provide a mean for the calculation of masses of bare particles.

In the field of cosmology the model – we have just seen an example – could shed a new light on the dynamics of the Big Bang.

As a final word it should be said that the present model is certainly not a theory of everything (a TOE). It simply adds a new picture to the already long list of cosmological models of the universe.

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Appendix A: Calculation of the “free functional” of a cosmic point

The functional of a CP ‘ i ’ is given by

$$H(\phi_i) = -\frac{J}{n} \sum_{\langle \alpha\mu, \beta\nu \rangle} \sigma_{i\alpha\mu} \sigma_{i\beta\nu}$$

where the sum is over all bit pairs of the point. Since

$$s_{i\mu} = \frac{1}{n_{i\mu}} \sum_{\alpha} \sigma_{i\mu\alpha}$$

one has

$$n_{i\mu} n_{i\nu} s_{i\mu} s_{i\nu} = \left(\sum_{\alpha} \sigma_{i\mu\alpha} \right) \left(\sum_{\beta} \sigma_{i\nu\beta} \right) = \sum_{\alpha\beta} (\sigma_{i\mu\alpha} \sigma_{i\nu\beta})$$

and

$$H(\phi_i) = -\frac{J}{2n} \sum_{\langle \mu\nu \rangle} n_{i\mu} n_{i\nu} s_{i\mu} s_{i\nu}.$$

The partition function is

$$Z_i = \sum_{\{\sigma_i\}} \exp(-bH(\phi_i))$$

where the sum is over all possible configurations of the cosmic point. The sum may be carried out in two steps:

$$Z_i = \sum_{\{s_{i\mu}\}} \exp\left(\frac{bJ}{2n} \sum_{\langle \mu\nu \rangle} n_{i\mu} n_{i\nu} s_{i\mu} s_{i\nu}\right) \sum_{\{\sigma_i\}} \sum_{s_{i\mu} \text{ given}} 1,$$

that is to say a sum over all possible sets of polarizations and, then, a sum over all bits states for a given set of polarizations. The last term

$$W = \sum_{\{\sigma\}, s_{i\mu} \text{ given}} 1$$

is purely combinatorial in nature. It is given by

$$W = \frac{n!}{\prod_{\mu} (n_{i\mu\uparrow}! n_{i\mu\downarrow}!)}$$

where

$$n_{i\mu\uparrow} = \frac{n_{i\mu}}{2} (1 + s_{i\mu}),$$

$$n_{i\mu\downarrow} = \frac{n_{i\mu}}{2} (1 - s_{i\mu}).$$

By using the Stirling formula one obtains

$$\begin{aligned} \text{Ln}(W) \cong & - \sum_{\mu} \left[\frac{n_{i\mu}}{2} ((1 + s_{i\mu}) \text{Ln}(1 + s_{i\mu}) \right. \\ & \left. + (1 - s_{i\mu}) \text{Ln}(1 - s_{i\mu})) \right] \end{aligned}$$

where a non-relevant constant has been skipped. The partition function reads

$$\begin{aligned} Z = \sum_{\{s_{\mu}\}} \exp & \left[\frac{bJ}{2n} \sum_{\mu\nu} n_{i\mu} n_{i\nu} s_{i\mu} s_{i\nu} \right. \\ & \left. - \sum_{\mu} \left[\frac{n_{i\mu}}{2} ((1 + s_{i\mu}) \text{Ln}(1 + s_{i\mu}) \right. \right. \\ & \left. \left. + (1 - s_{i\mu}) \text{Ln}(1 - s_{i\mu})) \right] \right]. \end{aligned}$$

It is given the following form,

$$Z_i = \sum_{\{s_{\mu}\}} \exp(-bF(\phi_i)),$$

a sum which, in the thermodynamic limit, reduces to one term, the term which minimizes F . The thermal averages of order parameters are then given by the order parameters which makes F minimum (the so-called saddle point method) and one has

$$\begin{aligned} F(\phi_i) = & \frac{-J}{2n} \sum_{\mu\nu} n_{i\mu} n_{i\nu} \langle s_{i\mu} \rangle \langle s_{i\nu} \rangle \\ & + \sum_{\mu} \frac{n_{i\mu}}{2b} ((1 + \langle s_{i\mu} \rangle) \text{Ln}(1 + \langle s_{i\mu} \rangle) \\ & + (1 - \langle s_{i\mu} \rangle) \text{Ln}(1 - \langle s_{i\mu} \rangle)). \end{aligned}$$

The realizable physical states are those that minimize F . If we consider the case where $d = 1$ that is to say if

$$n_{i\mu} = n\delta_{\mu 1}; \quad \langle s_{i\mu} \rangle = \varphi_i \delta_{\mu 1}$$

the free functional reduces to

$$\begin{aligned} F(\varphi_i) = & n \left(\frac{-J}{2} \varphi_i^2 + \frac{1}{b} \left[\left(\frac{1 + \varphi_i}{2} \right) \text{Ln}(1 + \varphi_i) \right. \right. \\ & \left. \left. + \left(\frac{1 - \varphi_i}{2} \right) \text{Ln}(1 - \varphi_i) \right] \right) \end{aligned}$$

Appendix B: Physics must be left invariant under the operations of the group of permutations of four objects

The permutation group S_4 of four objects has $4! = 24$ elements. The invariance of four dimensional matrices, such as G , under those transformations, requires the matrices to commute with the 24 matrices of permutation. An example of a permutation matrix is

$$\begin{pmatrix} . & . & . & 1 \\ 1 & . & . & . \\ . & . & 1 & . \\ . & 1 & . & . \end{pmatrix}$$

which is a four dimensional representation of permutation $(1234) \Rightarrow (2431)$. Let Γ_4 be this representation. Since S_4 has 5 classes there exist 5 irreducible representations which are

$$\Gamma_1, \Gamma_1^*, \Gamma_2, \Gamma_3, \Gamma_3^*$$

with orders 1, 1, 2, 3, and 3, respectively [11]. The table of characters of these representations is given in Table 1 and the table of characters of representation Γ_4 in Table 2.

Table 1. Table of characters of group S_4

classes	(1) : 1	(ab) : 6	(ab)(cd) : 3	(abc) : 8	(abcd) : 6
Γ_1	1	1	1	1	1
Γ_1^*	1	-1	1	1	-1
Γ_2	2	0	2	-1	0
Γ_3	3	1	-1	0	-1
Γ_3^*	3	-1	-1	0	1

Table 2. Table of characters of representation Γ_4

classes	(1) : 1	(ab) : 6	(ab)(cd) : 3	(abc) : 8	(abcd) : 6
Γ_4	4	2	0	1	0

From these two tables one finds the decomposition of Γ_4 into irreducible representations of S_4 as a direct diagonalization of G shows

$$\Gamma_4 = \Gamma_1 \oplus \Gamma_3 .$$

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